

BOUNDARY VALUE PROBLEMS FOR BLOCK-STRUCTURED DIFFERENTIAL SYSTEMS WITH NON-SEPARATED CONDITIONS

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ABSTRACT

Problems involving systems of ordinary differential equations with non-separated, non-linearized boundary conditions are boundary value problems. These problems are natural in large-scale, interconnecting and multi-stage dynamical models. This is especially difficult in situations where the governing equations have a block-structured form, which is indicative of interactions at the subsystem level and coupling at the global boundary. The paper is one which constructs a coherent operator-theoretic model of the analysis of a boundary value problem involving block-structured linear differential systems that are subject to the unseparated action of boundary constraints. General existence and uniqueness conditions are determined by setting up the problem in the form of differential and boundary operators and a fundamental matrix approach is applied. The resulting solvability criteria explicitly put into consideration global boundary coupling by an algebraic criterion of the fundamental matrix. Moreover, block sweep and transfer methods are studied and stability as well as convergence are determined assuming diagonal block dominance. A good example proves the generalizability of the theoretical findings to oscillatory block systems in coupled boundary conditioning. The suggested framework offers a strict mathematical basis of the systematic analysis and effective methodologies of solving big-scale boundary value problems in applied mathematics and engineering.

Keywords: *Boundary value problems, Block-structured differential systems, non-separated boundary conditions, Operator-theoretic methods, Existence and uniqueness, Block sweep methods.*

1. INTRODUCTION

Ordinary differential equations BVPs are a sub-discipline of mathematical analysis and applied modeling, of long interest in physics, engineering, and applied sciences. Classical theory the classical theory of BVP concerns mainly scalar equations or low-dimensional systems with separated boundary conditions, in

which conditions at each endpoint are independent of one another. In this context, established tools of analysis offer explicit solvability conditions, stability analysis and constructive solution.

Nevertheless, in many modern scientific and engineering systems, there are large-scale coupled systems, where the interactions between subsystems occur both at their internal dynamics and at their global boundary conditions. These systems are often the products of multi-stage process modeling, discretized space modeling, network modeling and hierarchical control architecture modeling. With increasing scale of dimensionality and coupling complexity, classical assumptions of separated boundary conditions can tend to become ineffective.

Such interactions naturally produce block structured differential systems, where the system matrix currently has a block decomposition which reflects subsystem organization. Besides internal coupling through off-diagonal blocks, the corresponding boundary conditions are often no longer separated, i.e., the solution value at one endpoint is contingent on what is occurring at several endpoints. This form of boundary coupling represents physical continuity constraints, feedback or conservation laws between separate components of the system. High-profile cases can be found in networked dynamical systems, fluid and aerodynamic models, multi-physics simulation and global-constrained control.

Although such models have increasingly become prevalent, the current literature is mainly biased towards numerical implementation and application specific research. The existence, uniqueness and structural solvability of block-structured systems in non-separated boundary conditions have been analyzed with rigorous analytical findings that are relatively few. Particularly, formulations and criteria of solvability that explicitly use the block structure of operators remain immature.

The paper intends to fill this gap by creating a single mathematical model of a system of non-separated boundary value problems of block-structured systems of ordinary differential equations with non-separated boundary conditions. The analysis of the operators using operator theoretic methods and block systems analysis proves the existence and uniqueness results of the general cases and lays a groundwork in the efficient methods of block-based solutions. The analysis is limited to linear systems, which forms a basic basis in nonlinear extensions, and in the development of stable numerical schemes of large-scale problems.

2. LITERATURE REVIEW

Ashrafova (2019) investigated a method of block transfer of conditions for systems of three-step discrete processes connected through boundary conditions. The study developed a mathematical approach to analyze and solve discrete dynamic systems by efficiently transferring boundary conditions across multiple steps. It highlighted that the proposed method improved computational efficiency and provided accurate solutions for complex discrete models, contributing to advancements in informatics and control systems analysis.

Alhawwary et al. (2020) examined the flow over the T106C turbine using Direct Numerical Simulation (DNS) and Large Eddy Simulation (LES) based on a high-order FR/CPR method. The study demonstrated that high-order numerical techniques significantly enhanced the accuracy of flow predictions, particularly in capturing turbulence and complex aerodynamic behaviors. It also emphasized the importance of advanced computational methods in improving the design and performance analysis of turbine systems.

Lienard and Boisard (2018) investigated the aerodynamics of the MEXICO rotor in axial flow, including the effects of boundary layer transition. The study revealed that accurately modeling boundary layer transitions was essential for predicting rotor performance and aerodynamic characteristics. It showed that incorporating these effects improved the reliability of simulations and provided deeper insights into wind turbine efficiency and design optimization.

Keye et al. (2017) presented the DLR results from the sixth AIAA computational fluid dynamics drag prediction workshop, focusing on the evaluation of drag prediction methods. The study analyzed various computational approaches and compared their accuracy in predicting aerodynamic drag. It highlighted the challenges associated with drag prediction and emphasized the need for improved numerical models and validation techniques to enhance the reliability of computational fluid dynamics simulations in aerospace engineering.

3. PRELIMINARIES AND PROBLEM FORMULATION

Let $[a, b] \subset \mathbb{R}$ be a compact interval. We consider vector-valued functions of the form

$$y(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_m(x) \end{bmatrix} \in \mathbb{R}^N,$$

where each component $y_i(x) \in \mathbb{R}^{n_i}$ represents the state of a distinct subsystem, and the total system dimension satisfies.

$$\sum_{i=1}^m n_i = N.$$

This decomposition is a reflection of the inner architecture of the system, which makes it possible to describe explicitly the interaction between subsystems. To be more precise and consistent, the main symbols and operators that are going to be used in the paper are summarized in Table 1.

Table 1: Notation and Symbols Used

Symbol	Description
$y(x)$	State vector of the system
$y_i(x)$	State vector of the i -th subsystem
$A(x)$	Block-structured system matrix
B_a, B_b	Boundary condition matrices
$\Phi(x)$	Fundamental matrix solution
$\mathcal{L}$	Differential operator
$\mathcal{B}$	Boundary operator

We study block-structured systems of ordinary differential equations of the form

$$\frac{dy}{dx} = A(x)y(x) + f(x), x \in [a, b], \quad (3.1)$$

and the coefficient matrix $A(x) \in \mathbb{R}^{N \times N}$ in which there is a block decomposition that is consistent with the partition of $y(x)$. The diagonal blocks of $A(x)$ characterize the inner workings of single subsystems, whereas the off-diagonal blocks represent the effect of the connection between the various subsystems. The $f(x)$ is inhomogeneous, which is supposed to be a continuous term on $[a, b]$, so that the initial value problem is well-posed.

Non-separated boundary conditions of the form are added to the system.

$$B_a y(a) + B_b y(b) = c, \quad (3.2)$$

where $B_a, B_b \in \mathbb{R}^{N \times N}$ are constant matrices and $c \in \mathbb{R}^N$. As opposed to classical separated boundary conditions, in such restrictions, the value of solutions on both ends is involved in a fashion that provides global coupling between subsystem components.

The theory has a number of significant classes of boundary conditions as special cases such as periodic, mixed, and non-local constraints. In turn, it offers a natural and flexible mathematical framework of interconnected, multi-stage, and network-structured dynamical systems experienced in applied mathematics, engineering, and control theory.

Table 2 summarises the various types of the boundary conditions that are included in the proposed framework and their mathematical representations.

Table 2: Types of Boundary Conditions Covered

Boundary Type	Mathematical Form	Covered in This Framework
Separated	$\gamma(a) = \alpha, \gamma(b) = \beta$	Yes
Mixed	$\gamma(a) = \alpha, C\gamma(b) = \beta$	Yes
Periodic	$y(a) = y(b)$	Yes

Non-separated	$B_a y(a) + B_b y(b) = c$	Yes
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4. OPERATOR-THEORETIC FRAMEWORK

In order to study the boundary value problem in one mathematical framework, we propose an operator-theoretic formulation which unites the differential equation and boundary conditions. Specify the linear differentiating operator.

$$\mathcal{L}y = \frac{dy}{dx} - A(x)y,$$

with domain $C^1([a, b], \mathbb{R}^N)$, and the associated boundary operator

$$\mathcal{B}y = B_a y(a) + B_b y(b).$$

With the help of these operators, the operator system can be expressed as a compact form of the boundary value problem (3.1) (3.2).

$$\mathcal{L}y = f, \mathcal{B}y = c.$$

This formulation captures both the differential dynamics and the boundary constraints within a unified linear framework, which makes possible the use of functional analytic methods such as solvability analysis, null space characterization, and invertibility of boundary operators. Specifically, the operator formulation enables a boundary value problem to be considered as a constrained operator equation in an appropriate function space, which helps to illuminate the role of boundary coupling in determining well-posedness.

Further, the coefficient matrix $A(x)$ has a block structure that is inherited by the operator $\mathcal{L}$, and it maintains interaction on a subsystem level in the analytical structure. This structural inheritance enables block-by-block analysis as well as offering a theoretical basis of organized solution methods, such as block sweep and block transfer methods, which are particularly useful to large-scale systems with non-separated boundary conditions.

5. EXISTENCE AND UNIQUENESS RESULTS

In order to come up with strict solvability conditions of the studied boundary value problem, the existence and uniqueness of solutions is analysed in this part through the use of a fundamental matrix approach. It goes on to discuss the representation of a general solution to the block-structured system, the algebraic solvability condition that arises as a result of the non-separated boundary constraints and the existence and uniqueness theorem that defines well-posedness of the problem.

5.1 Fundamental Matrix and Solution Representation

Let $\Phi(x)$ denote a fundamental matrix solution of the associated homogeneous system

$$\frac{dy}{dx} = A(x)y(x),$$

that is, $\Phi(x)$ satisfies the matrix differential equation $\Phi'(x) = A(x)\Phi(x)$ with $\det \Phi(x) \neq 0$ for all $x \in [a, b]$. The existence and continuity of such a fundamental matrix follow directly from the continuity of the coefficient matrix $A(x)$ on the compact interval $[a, b]$.

Using the formula of variation of constants, one can obtain any solution of the nonhomogeneous system (3.1) in the form.

$$y(x) = \Phi(x)k + \int_a^x \Phi(x) \Phi^{-1}(s)f(s) ds,$$

in which $k \in \mathbb{R}^N$ is a constant vector that depends on the boundary conditions uniquely. In this form it is easy to distinguish the effect of the homogeneous dynamics, represented in the term $\Phi(x)k$, and the contribution of the forcing term $f(x)$, represented by the integral. Consequently, the effects of boundary constraints and inhomog get affected.

5.2 Solvability Condition for Non-Separated Boundary Constraints

Here we replace the general solution representation that was obtained in Section 5.1 in the non-separated boundary condition.

$$B_a y(a) + B_b y(b) = c$$

yields a linear algebraic system for the unknown constant vector $k \in \mathbb{R}^N$ of the form

$$(B_a \Phi(a) + B_b \Phi(b))k = c - B_b \int_a^b \Phi(b) \Phi^{-1}(s) f(s) ds.$$

This relationship indicates that the solvability of the boundary value problem is determined by the invertibility of the matrix $B_a \Phi(a) + B_b \Phi(b)$, which represents the system dynamics and global boundary coupling. Unlike the classical separated boundary conditions, in which the boundary constraints are independent at each endpoint, the current formulation involves a coupling matrix that represents the interactions over the entire interval.

The matrix $B_a \Phi(a) + B_b \Phi(b)$ therefore has a similar role as the boundary determinant in traditional two point boundary value problems but in a generalized form which is explicitly sensitive to block structure and to non-separated boundary conditions.

5.3 Existence and Uniqueness Theorem

Theorem 5.1.

If

$$\det(B_a \Phi(a) + B_b \Phi(b)) \neq 0,$$

then the problem of boundary values (3.1) and (3.2) has a solutions on $[a,b]$ which is unique.

Proof.

The no singularity of the matrix $B_a \Phi(a) + B_b \Phi(b)$ ensured that the linear algebraic system obtained in Section 5.2 has a unique solution to the constant vector $k \in \mathbb{R}^N$. Replacing this specially chosen vector into the general solution form in Section 5.1 results in a function $y(x)$ which is a solution to the differential

equation (3.1) at all x in $[a, b]$. Furthermore, by definition this function fulfils also the non-separated boundary condition (3.2). Therefore, the boundary value problem has a unique solution on $[a, b]$.

Remark.

Generalizing classical existence and uniqueness results of two-point problems with separated boundary conditions and with two-point boundary value problems, Theorem 5.1 applies to the problem of block-structured systems with non-separated boundary constraints. In particular, the solvability condition explicitly incorporates global boundary coupling through the matrix $B_a\Phi(a) + B_b\Phi(b)$, thereby extending standard boundary determinant criteria to a broader and more structured class of dynamical systems.

6. BLOCK SWEEP AND TRANSFER TECHNIQUES

Block sweep and transfer methods offer an effective way of solving block-structured problems of the boundary value by capitalizing on subsystem interactions. Unlike the component-based techniques, these methods directly act on blocks, which is especially appropriate to large-scale systems with global coupling at boundary. In the sections below, the conceptual underpinning, stability, and convergence of the block sweep technique are described.

6.1 Block Transfer Representation

In block-organized systems, block transfer relations can be used to describe the solution of a given subsystem in terms of neighbouring blocks. These relations define the flow of boundary information through blocks and can be formulated as linear operators on a block boundary data. The original coupled system can be simplified to a series of lower-dimensional block problems, based on such representations, yet maintaining the global structure imposed on it by non-separated boundary conditions.

6.2 Stability under Diagonal Block Dominance

In the event of the block coefficient matrix having a diagonal dominance, the corresponding block transfer matrices are uniformly bounded on the interval $[a, b]$. This boundedness is what guarantees that perturbations injected in one block do not grow exponentially as they pass through the system. As a result of these, numerical stability is preserved notwithstanding the presence of strong inter-block coupling and global boundary constraints.

6.3 Convergence and Solvability Preservation

Theorem 6.1.

For block-structured systems with diagonally dominant block coefficients, the block sweep algorithm is convergent and maintains solvability for non-separated boundary conditions.

Proof.

Diagonal block dominance suggests that the block transfer operators introduce contractions based on an appropriate operator norm. The resultant block-wise recursion consequently converges. By the block sweep procedure, the boundary solvability conditions in Section 5 are satisfied hence convergence of the recursion guarantees both stability and accuracy of the solution.

7. ILLUSTRATIVE EXAMPLES AND REMARKS

Example 7.1

Take a two block system of ordinary differential equations that is represented by

$$\frac{d}{dx} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

where I denotes the identity matrix of the right dimension. A prominent skew-symmetric off-diagonal block structure reveals a common feature of the system matrix; that of conservative and oscillatory dynamics. This sort of system frequently occurs in coupled mechanical oscillators, Hamiltonian formulations and discrete wave propagation models.

The system is supplemented with the non-separated boundary condition

$$y_1(a) + y_2(b) = 0,$$

that couples the components of the solution that are considered at opposite endpoints of the interval. This form of boundary condition cannot be simplified to separated behavior at a and b as independent variables and thus is an exemplary example of the global boundary coupling of interest in this paper.

In the current system, a basic matrix $\Phi(x)$ could be explicitly built based on using matrix exponentials and it could be confirmed that the corresponding boundary matrix $B_a\Phi(a) + B_b\Phi(b)$ is non-singular. Consequently, the solvability statement of Theorem 5.1 is met and has a solution on the interval $[a, b]$. This example thus illustrates how the abstract solvability criteria developed in the previous sections can be applied to concrete block-structured systems with non-separated boundary conditions.

Remark 7.1

The coupled boundary conditions of the form discussed above occur naturally in a variety of problems, such as oscillatory systems with phase or symmetry constraints, discretized models of flow and transport problems where the continuity conditions relate nodes that are spatially separated, and network dynamical systems that are governed by global conservation or synchronization constraints. This example illustrates that even in the case of strong coupling between blocks and global boundary constraints, block-structured systems can be analyzed systematically using the above operator-theoretic approach. In this regard, this example not only illustrates the above theoretical developments but also points to the potential applicability of the approach to other structured dynamical systems in applied mathematics and engineering.

8. CONCLUSION

This research produced a common operator-theoretic treatment of boundary value problems of block-structured systems of non-separated boundary conditions of ordinary differential equations. General existence and uniqueness results were proven using a fundamental matrix method, by expressly adding block structure and global coupling of the boundary to the analytical formulation, to a generalization of classical solvability conditions of separated boundary value problems. The analysis also proved that block sweep and transfer methods offer consistent and converging solution measures to large-scale systems in the case of diagonal block dominance, without compromising solvability and enhanced computational performance. A practical example was used to demonstrate the practical importance of the theoretical findings and accentuate the relevance of non-separated boundary constraints in structured dynamical models. Altogether, the given framework has a strict mathematical background of the analysis and solution of interconnected differential systems and can be discussed as the basis of the further development of the approach to the nonlinear problems, the interaction between time-dependent boundaries, and the improved block-based numerical techniques.

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